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Breadth of Data Dispersion and Number of Variables as Sources of Measurement Effects on Factor Variances in Confirmatory Factor Analysis

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Abstract

The present study investigates how simple measurement effects influence factor variances in CFA. The considered measurement effects refer to a) deviations from the expected dispersion of data and b) the number of manifest variables (e.g., items of a scale) loading on a factor while the underlying data structure is kept constant. In this investigation, the factor variance is conceptualized as the scaled variance parameter of the model-implied covariance matrix of the Maximum Likelihood approach. The results of model analyses and a simulation study revealed that the modification of the breadth of data dispersion and the number of manifest variables systematically influenced scaled factor variances despite the constancy of the latent structure. Furthermore, the results revealed that the effects on the factor variance could be eliminated by either estimating factor variances using standardized data or by standardizing estimated factor loadings before their conversion into factor variances.

Keywords: factor variance, dispersion, variable number, scaling, confirmatory factor analysis

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Factor variances have so far not played a major role in the investigation of the structure of data since the customary measurement model of confirmatory factor analysis (CFA) is a one-factor model (Brown, 2015; Graham, 2006). In structural investigations using this model, fit indices provide sufficient information for evaluating the model. However, the situation is about to change since CFA models including two or more factors have become available and popular such as the multitrait-multimethod models (Byrne, 2016) and the bifactor model (Reise, 2012). Although fit indices still provide important information for evaluating models including two or more factors, they are not especially well suited for determining for how much of the systematic variation of data individual factors account. One-factor models and models including two factors can be compared by using differences between χ^2s and some fit indices as, for example, Comparative Fit Index (CFI) and Root Mean Square Error of Approximation (RMSEA) (Cheung & Rensvold, 2002). But, their utility for quantifying the individual factors' contribution is limited since fit statistics reflect model-data fit in the first place but not the systematic variation for which a factor accounts.

The factor variance reflects the amount of systematic variation of data for which a factor accounts if the factor variance is defined in the traditional way as sums of squares of factor loadings. This may not be immediately obvious since factor loadings are not known as measures of the spread of scores, such as variances (Vogt & Johnson, 2015). However, it is established that squares of factor loadings reflect the proportion of the variance of a manifest variable that this variable has in common with the factor. So, factor variances are rooted in the variances of manifest variables which reflect the spread of participants' scores. Although such factor variances and factor loadings are closely linked, the attention in factor analysis so far has been almost exclusively on factor loadings (Widaman, 2018).

The disregard of factor variances in CFA may be due to the way of its integration into the model of the covariance matrix as a variance parameter (Gumedze & Dunne, 2011; Jöreskog, 1970). Variance parameters serving as factor variances need additional specification to serve well for this purpose. There are three ways of specifications that are known as scaling methods (Little, Slegers, & Card., 2006). First, there is reference-group scaling meaning that the variance parameter is set equal to one. It is mostly used for investigating the difference between the factor variances of the reference group and other groups. Second, there is the marker-variable method that requires fixing one factor loading to the value of one while the other factor loadings and the variance parameter are estimated. Though, in this case, the variance parameter is estimated, its estimate heavily depends on the selected marker variable, that is, on the variable with the fixed factor loading (e.g., Gonzales & Griffin, 2001; Steiger, 2002). Third, there is the criterion-based method of which the main characteristic is a criterion number. It requires the standardization of the factor loadings in such a way that they either add up directly to the criterion number (Little et al., 2006) or that the squares of the factor loadings add up to the criterion number (Schweizer, 2011). Equivalence of the estimate of the factor variance and the sum of squared factor loadings is reached by selecting one as the criterion number for the sum of squared factor loadings (Schweizer & Troche, 2019).

Appropriately scaled factor variances yield estimates of factor variances that are in line with the traditional definition as the sum of squared factor loadings. But, it is still possible that a distortion originates in the variances of the scores giving rise to the manifest variables that means a deviation from what is expected. Such distortion can impair their use for representing the amount of systematic variation for which the factors account and for comparing different factors by their factor variances. The comparison of variances is of interest as part of invariance analysis (Schmitt & Kuljanin, 2008; Thompson, 2016) across different populations or different measurement procedures representing the same latent construct.

Distortion of factor variances can occur in the sense of *squeezing* or *stretching* of the dispersions of data represented by manifest variables from which factors are extracted (Vogt & Johnson, 2015). Such distortion can happen in the context of measurement and when participants' attributes are transformed into numbers. First, the situation of measurement can influence how participants respond to items. For example, participants may feel encouraged to express extreme positions in one situation or to the contrary in another one. Second, changes of sample characteristics can cause deviations. While one sample may show a range restriction regarding the property of interest, another sample may show a wide dispersion. Third, there may be procedural specificity due to the specific mode of assessment as, for example, is obvious in attention assessment using a paper-and-pencil test on one hand and a psycho-motor task on the other hand. Further, ceiling and bottom effects can exert an influence on the dispersion. The use of range-restricted response formats is especially susceptible to this type of deviation. Next, a sample may include outliers that heavily influence the dispersion of data. Finally, we like to point to possible modifications of the dispersion due to changes in the coding system and eventually necessary non-linear transformations.

A distortion of the dispersion can mean squeezing or stretching in the sense of a deviation from what is considered as the latent dispersion. In this way increased or decreased variances may lead to incorrect conclusions regarding the factors' accounts of systematic variation of data. Systematic differences in variation that are characteristic of types of instruments can become apparent as method factors in CFA based on, for example, a multitrait-multimethod (MTMM) design (Byrne, 2016). Such MTMM investigations can disclose specificities of information processing due to variations of instruments, observers, and occasions (Biesanz & West, 2004).

Furthermore, the reliability of measurement scales is well-known to depend on the number of items they are composed of. This dependency suggests another source of influence on factor variances. Increasing the number of parallel items is known to increase the reliability of the corresponding scale. The Spearman-Brown formula enables the estimation of the consequence of such an increase (Greer & Liu, 2016). The possibility to enlarge the reliability by increasing the number of items suggests incompleteness in capturing relevant systematic variation of data. An implication is that more items may lead to larger factor variances. Another implication is that the switch from a substantial to a zero factor loading in a specific sample in the sense of differential item functioning (French & Holmes, 2016) means a decrease of factor variance in the specific sample.

While the idea of factor variances rooted in the variances of manifest variables and of moderation by factor loadings is well established, the suspected influences of variance sizes of manifest variables and variable numbers on factor variances still requires systematic study. This is especially important because the effect of the underlying structure of data also needs to be considered. In the following the arguments are presented in a formal way and a simulation study is reported to strengthen the considerations by empirical evidence.

The Scaling of Variances of Latent Variables

This section introduces the scaling of factor variances. It is necessary since scaling establishes the stable ground for further investigations focusing on the dependency of factor variances on the dispersion of data and the number of variables. Scaling occurs within the framework of the model of the covariance matrix (Gumedze et al., 2011; Jöreskog, 1970) associated with the CFA measurement models (Graham, 2006). In the case that the measurement model includes one factor (= latent variable), it is the model of the $p \times p$ covariance matrix, $\mathbf{\Sigma}$ in $\Re^{p\times p}$, that is defined as

$$\mathbf{\Sigma} = \boldsymbol{\lambda}\phi\boldsymbol{\lambda}' + \boldsymbol{\theta} \quad (1)$$

where $\boldsymbol{\lambda}$ is the $p \times 1$ vector of factor loadings, $\phi \in \Re^{\geq 0}$ the variance parameter, and $\boldsymbol{\theta}$ the $p \times p$ diagonal matrix of residuals. A multiplicative relationship between factor loadings and the variance parameter characterizes this model. This multiplicative relationship gives rise to what is called a constancy framework (Schweizer, Troche, & DiStefano, 2019). Given $p \times 1$ vector $\boldsymbol{\lambda}$, scalar ϕ and constant $c > 0$, this framework states that $\boldsymbol{\lambda}\phi\boldsymbol{\lambda}' = \boldsymbol{\lambda}^*\phi^*\boldsymbol{\lambda}^{*\prime}$ if $\boldsymbol{\lambda}^* = c\boldsymbol{\lambda}$ and $\phi^* = 1/c^2 \times \phi$. This means that $\mathbf{\Sigma}$ stays constant while there is a systematic change from the combination of $\boldsymbol{\lambda}$ and ϕ to the combination of $\boldsymbol{\lambda}^*$ and ϕ^*. Estimation of $\boldsymbol{\lambda}$ and ϕ must precede the computation of $\boldsymbol{\lambda}^*$ and ϕ^*. All available scaling methods can be described within this framework.

The constancy framework suggests that parameter estimation does not guarantee that the variance parameter (ϕ) is in line with the traditional definition of the factor variance as the sum of squared factor loadings although this is desirable. To achieve estimates of variance parameters that are in line with the traditional definition, we start with the formal representation of the traditional definition: let $\varphi \in \Re^{>0}$ represent the factor variance and $\lambda_1, \ldots, \lambda_p$ the factor loadings. Then, the traditional definition of the factor variance is given by

$$\varphi = \sum_i \lambda_i^2 = \text{trace}(\boldsymbol{\lambda\lambda}') \quad . \tag{2}$$

The latter part of Equation 2 lacks a variance parameter that is necessary to associate the argument of the trace to the first part of the right-hand side of Equation 1. But, if the variance parameter is scaled according to the reference-group method ($\phi = 1$), there is still correspondence to the sum of squared factor loadings:

$$\text{trace}(\boldsymbol{\lambda}\phi\boldsymbol{\lambda}') = \sum_i \lambda_i^2 \tag{3}$$

Finally, the traditional definition of the factor variance is characterizing the product of the vectors of factor loadings and the variance parameter that is also part of the model of the covariance matrix (Equation 1):

$$\varphi = \sum_i \lambda_i^2 = \text{trace}(\boldsymbol{\lambda}\phi\boldsymbol{\lambda}') . \tag{4}$$

Thus, the input to the trace operator corresponds to the first component of the right-hand part of Equation 1. This means that it is possible to obtain the factor variance according to the traditional definition from information provided by CFA.

However, Equation 4 enables the estimation of φ in an indirect way only. To estimate φ directly that means parameter estimation so that $\varphi = \phi$, an additional step is necessary that makes use of the constancy framework. This step proceeds from given estimates of factor loadings, λ_i ($i = 1,\ldots, p$), that is, from estimates obtained on the basis of reference-group scaling ($\phi = 1$). Subsequently, scalar $c > 0$ is computed so that

$$1 = \sum_i (c\lambda_i^2) = \sum_i \lambda_i^{*2} \tag{5}$$

(that means criterion-based scaling) with $\lambda_i^{*2} = (c\lambda_i)^2$. The constancy framework requires introducing modified factor loadings in the right-hand part of Equation 4. These factor loadings have to be complemented by a modified variance parameter: $\phi^* = 1/c^2\phi$. Next, Equation 4 can be re-written as

$$\varphi = \text{trace}(\boldsymbol{\lambda}^* \phi^* \boldsymbol{\lambda}^{*\prime}). \tag{6}$$

Since ϕ^* is a scalar and $\boldsymbol{\lambda\lambda}' = 1$ (Equation 5), equality of φ and ϕ^* is reached:

$$\varphi = \text{trace}(\boldsymbol{\lambda}^* \phi^* \boldsymbol{\lambda}^{*\prime}) = \phi^* . \tag{7}$$

In sum, starting with estimating factor loadings under the condition of $\phi = 1$ and proceeding with estimating ϕ^* under the condition that factor loadings are fixed to correspond to λ_i^* ($i = 1,\ldots, p$) finally yields the factor variance, ϕ^*, that is in line with its traditional definition as the sum of squared factor loadings (φ).

The Effect of Modifications of the Data Dispersion

This section addresses the consequences of influencing the dispersion of data in the sense of *squeezing* or *stretching*. This means that it is investigated whether and how squeezing or stretching the dispersion of manifest variables influences factor variances.

Let s_{ii} (i = 1,…,p) be the variance of the ith manifest variable taken from the main diagonal of the empirical $p \times p$ covariance matrix, **S**, and σ_{ii} (i = 1,…,p) an entry of the main diagonal of the corresponding $p \times p$ model-implied covariance matrix, **Σ**, that is the result of fitting **Σ** to **S**, by an established estimation method so that

$$s_{ii} \approx \sigma_{ii} \quad \text{with} \quad \sigma_{ii} = (\lambda_i \phi \lambda_i') + \theta_{ii} \tag{8}$$

The right-hand part reveals the assumed underlying structure according to Equation 1. An exact correspondence of s_{ii} and σ_{ii} is not assumed because of random influences that can cause deviation between s_{ii} and σ_{ii} impairing model fit in CFA.

Both squeezing and stretching that are represented by $a \in \Re^+$ (stretching: $a > 1$ and squeezing: $1 > a > 0$) can be expected to modify s_{ii} and σ_{ii} in the following way:

$$as_{ii} \approx a\sigma_{ii} \quad \text{with} \quad a\sigma_{ii} = a[(\lambda_i \phi \lambda_i') + \theta_{ii}] = a(\lambda_i \phi \lambda_i') + a\theta_{ii} \tag{9}$$

The further reasoning focuses the consequence of squeezing and stretching (a) for the factor variance (φ). Implicitly the assumption is made that a is the same for all manifest variables. The reasoning concentrates on the right-hand part of Equation 9. To achieve factor loadings that are in line with the definition of factor variances as the sum of squared factor loadings, ϕ needs to be fixed to one so that the factor loadings can be estimated. This requires associating scalar a with the factor loadings:

$$a(\lambda_i \phi \lambda_i') + a\theta_{ii} = (\sqrt{a}\lambda_i \phi \sqrt{a}\lambda_i') + a\theta_{ii} \tag{10}$$

This means that the expected result of estimating factor loadings under the condition of $\phi = 1$ represented by $\hat{\lambda}_i$ (i = 1,…, p) is related to the original factor loadings in the following way:

$$\hat{\lambda}_i = \sqrt{a}\lambda_i \tag{11}$$

This means that the factor variance reflecting squeezing or stretching, $\varphi_{S\&S}$, can be detailed as the sum of squared original factor loadings multiplied by a:

$$\varphi_{\mathrm{S\&S}} = \sum_i \hat{\lambda}_i^2 = a \sum_i \lambda_i^2 \tag{12}$$

Furthermore, it is possible to relate the factor variance of squeezed and stretched data, $\varphi_{\mathrm{S\&S}}$, to the factor variance of the original data, $\varphi_{\mathrm{original}}$. This just requires replacement of the sum of the right-hand part according to Equation 2:

$$\varphi_{\mathrm{S\&S}} = a\varphi_{\mathrm{original}} \tag{13}$$

This result suggests that there is either an increase or a decrease in factor variance that is proportional to the effect of squeezing or stretching as it is represented by *a*. The precondition for the validity of this result is that *a* is the same for all manifest variables.

The effect of the number of items

In this section the dependency of the factor variance on the number of items is investigated since data can show differential item functioning (French & Holmes, 2016) that means that some manifest variables may in some samples not load on the factor, as is expected. The reasoning proceeds from the assumption that all manifest variables show the same variance. Additionally, it makes use of results achieved in investigating the effects of squeezing and stretching the variances of manifest variables on the factor variance.

The reasoning starts from the situation of a number of manifest variables loading on the same factor that is modified by adding some equivalent manifest variables. To formalize this situation, assume a set of *m* manifest variables that is increased by *n* additional manifest variables loading on the same factor. Equation 2 suggests the summation of the squared factor loadings to arrive at the factor variance, $\varphi_{\mathrm{m+n}}$:

$$\varphi_{\mathrm{m+n}} = \sum_i^{m+n} \lambda_i^2 = \sum_i^{m} \lambda_i^2 + \sum_i^{n} \lambda_i^2 \tag{14}$$

In line with the reasoning in the previous section, we switch to an alternative way of describing the change from *m* to *m* + *n* manifest variables. This requires the transformation of the sum (*m* + *n*) into scalar $a \in \Re^+$:

$$a = (m+n)/m\,. \tag{15}$$

Scalar *a* enables treating an increase in number of manifest variables as treating stretching and treating a decrease in number as treating squeezing (see Equations 10 to 12). Thus, the effect of adding or removing manifest variables on the factor variance parallels the effect of modifying dispersion on the factor variance. Consequently, the conclusion from the previous section can be adopted: scalar *a* informs on how the original factor variance relates to the factor variance after the change of the number of manifest variables.

Standardization for Controlling Differences in Dispersion

This section addresses possible problems in comparing factor variances of factors extracted from different data sets because of differences in manifest variables' variances. Such differences can mean distortion leading to incorrect conclusions. Standardization focusing dispersion can prevent such distortion.

Given scalar $a \in \Re^+$ measuring the distortion of dispersion and $\hat{\lambda}_i \in \Re^+$ ($i = 1,\dots, p$) representing the distorted *i*th factor loading. The modified version of Equation 11 enables the achievement of dispersion-standardized factor loading λ_i. It is just necessary to shift the square root of scalar *a* from one side of the equality sign to the other:

$$\lambda_i = \frac{1}{\sqrt{a}} \hat{\lambda}_i \qquad (16)$$

so that

$$\varphi = \sum_i \left(\frac{1}{\sqrt{a}} \hat{\lambda}_i \right)^2 \qquad (17)$$

The validity of Equation 17 depends on whether scalar *a* correctly reflects the deviation of the variances of manifest variables from 1.00 as standard.

The Simulation Study

A simulation study investigated the influence of the dispersion of data and the number of manifest variables on the factor variance using simulated data. There was no separate treatment of the effects of dispersion and of the number of variables on factor variances since the analysis of their underlying structures already revealed that they posed the same problem (Equations 13 and 15). Simulated data with a known underlying structure were created for this purpose. The generated data were continuous and followed a normal distribution. They were arranged as matrices with different numbers of columns, and the entries of the columns were modified to show different degrees of dispersion. The factor variance was estimated by a one-factor CFA model that provided scaled variance parameters (see Equation 7). Proceeding from the results of analyzing the effects of changing the dispersion of data and number of variables on factor variances, the main aim was to check whether the expected effects would hold when investigating simulated data. Furthermore, we probed the above-explained possibility of controlling for the influence of dispersion by standardization.

Method

Design. The study design included the dispersion of data as the first independent variable. Three levels of dispersion were realized (s^2 = .25, 1.0, 4.0) that could be perceived as outcomes of squeezing or stretching. The number of manifest variables served as the second independent variable of the design. There were two levels: 10 and 20 manifest variables. Furthermore, there was a control variable regarding the underlying structure (influence of source) with three expected values for factor loadings (weak=.35, medium =.50, strong =.65). The dependent variables of the design were the sizes of factor variances and factor loadings.

Data generation. Data matrices included 500 rows and either 10 or 20 columns and were generated by means of three 10×10 relational patterns in the first case and three 20×20 relational patterns otherwise. The off-diagonal entries of these patterns were constructed to be reproducible by factor loadings of 0.35, 0.50 and 0.65 of a one-factor CFA model; we referred to them as weak, medium, and strong versions, respectively. The diagonal entries of the relational patterns were set equal to one. Each one of the six relational patterns served the generation of 500 matrices of continuous and normally distributed random data [N(0,1)] using PRELIS (Jöreskog & Sörbom, 2001).

To obtain different degrees of dispersion, the variances of the columns of the data matrices were modified. In one set of matrices the mean-centered data were multiplied by 0.25 and in another set by 4.00 so that there were three types of dispersion (variances of 0.25, 1.00 and 4.00). The variance of 1.00 was considered as outset (i.e., no distortion of dispersion).

Model. The confirmatory factor model included one factor (= latent variable). The factor was designed to capture the systematic variation due to the latent source of responding. Furthermore, there were either 10 or 20 manifest variables. Because of the way of data generation, equal-sized factor loadings were expected. To enable the estimation of scaled variances, the factor loadings on the factor were constrained. Values satisfying the following equation were assigned to the factor loadings:

$$1 = \sum_i \lambda_i^{\,2}$$

while the variance parameter of the latent variable was set free. Note. Equality of the variances of manifest variables is no precondition for scaling.

Statistical Investigation. Parameter estimation was conducted using maximum likelihood estimation (Schweizer et al., 2023). Covariances served as input to CFA. The variance parameter of the model served the estimation of the scaled factor variance. The computations were conducted using LISREL software (Jöreskog & Sörbom, 2006).

Average factor loadings and factor variances for different degrees of dispersion and different numbers of variables were calculated for comparing them with each other and with expected values. Furthermore, it was investigated to what extent factor variances and factor loadings reflected the influence of the latent source.

Results

Results regarding the factor variance. The variances of the latent variable achieved in investigating data are reported in Table 1.

The first to third columns of Table 1 inform about data characteristics due to data generation. The first column (dispersion) includes information on the variance of the manifest variables, the second column (number) on the number of manifest variables and the third column (influence of source) on the control variable that informs about the latent source. Next, there is the column providing what was expected regarding the factor variance. This column is subdivided into two parts. The first part gives the expectations according to Equation 2 and the second one according to Equation 13. Finally, there is the column with the means of the estimated factor variances. The standard deviation is added in parentheses.

Table 1

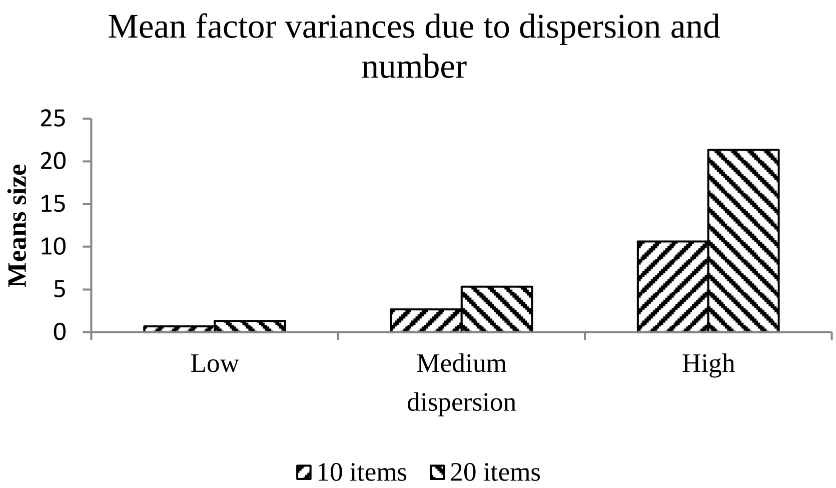
Means and Standard Deviations (in Parentheses) of Estimated Factor Variances ϕ_{sc} for Degrees of Dispersion, Numbers of Manifest Variables and Influences of latent Source (N = 500 Data Matrices)

Dispersion	Items	Influence of source	Expected variance φ_{λ}	Expected variance $\varphi_{a,\lambda}$	Estimated variance ϕ (SD)
0.25	10	Weak	1.22	0.31	0.31 (0.03)
1.00	10	Weak	1.22	1.22	1.23 (0.14)
4.00	10	Weak	1.22	4.90	4.92 (0.55)
0.25	10	Medium	2.50	0.62	0.63 (0.05
1.00	10	Medium	2.50	2.50	2.51 (0.22)
4.00	10	Medium	2.50	10.00	10.03 (0.87)
0.25	10	Strong	4.22	1.06	1.06 (0.08)
1.00	10	Strong	4.22	4.22	4.23 (0.33)
4.00	10	Strong	4.22	16.90	16.92 (1.33)
0.25	20	Weak	2.45	0.61	0.62 (0.06)
1.00	20	Weak	2.45	2.45	2.47 (0.22)
4.00	20	Weak	2.45	9.80	9.90 (0.88)
0.25	20	Medium	5.00	1.25	1.26 (0.10)
1.00	20	Medium	5.0	5.00	5.04 (0.39)
4.00	20	Medium	5.00	20.00	20.15 (1.54)
0.25	20	Strong	8.45	2.11	2.12 (0.15)
1.00	20	Strong	8.45	8.45	8.48 (0.61)
4.00	20	Strong	8.45	33.80	33.97 (2.46)

The results showed good correspondence between expectations and observations (last two columns) when the dispersion was 1.00. When the dispersion was either 0.25 or 4.00 and the expectation was based on Equation 2 that reflected the traditional definition of the factor variance (first part), there was a large discrepancy. In contrast, in expectations according to Equation 13 (second part) there was a good degree of correspondence. The results regarding the number of manifest variables also suggested an effect. The switch from 10 to 20 manifest variables led to an increase of the factor variance by 100 percent. Furthermore, there was an effect of the control variable (influence of source). The factor variance increased from weak to medium and further on to strong under the condition of a constant number of manifest variables.

The sizes of the effects of the independent and control variables on factor variances are illustrated by bars in Figure 1.

A



B

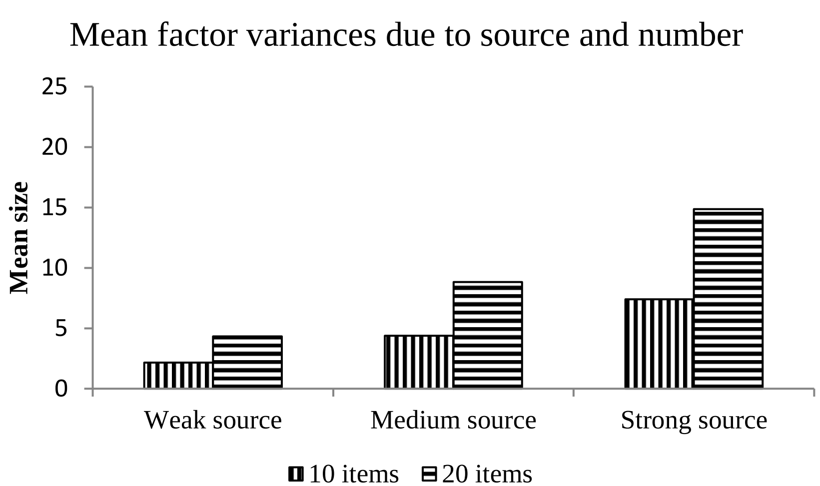


Figure 1.

Illustration of the mean factor variances due to dispersion while controlling for number (A) and of the influence of source while controlling for number (B).

The bars of Figure 1A represent the effects of the levels of dispersion in combination with the levels of numbers of manifest variables. Obviously, there was a non-linear increase from weak dispersion to strong dispersion. In contrast, the increase in size from 10 to 20 manifest variables is always 100 percent. The bars of Figure 1B depict the increases due to the modifications of the influence of the latent source and the levels of numbers of manifest variables. The bars suggest linear increases for both the influence of the latent source and the numbers of manifest variables.

In sum, the results corroborated the expected influence of the dispersion of data and of the number of manifest variables on the factor variance.

Table 2

Means of Estimated Factor Loadings for Degrees of Dispersion, Numbers of Manifest Variables and Influences of Latent Source Without (λ_n) and With Standardization ($\lambda_{s,n}$) (N = 500 Data Matrices)

Dispersion	Influence of source	Non-standardized		Standardized	
		$\lambda_{n=10}$	$\lambda_{n=20}$	$\lambda_{s,n=10}$	$\lambda_{s,n=20}$
0.25	Weak	0.17	0.17	0.35	0.35
1.00	Weak	0.35	0.35	0.35	0.35
4.00	Weak	0.70	0.70	0.35	0.35
0.25	Medium	0.25	0.25	0.50	0.50
1.00	Medium	0.50	0.50	0.50	0.50
4.00	Medium	1.00	1.00	0.50	0.50
0.25	Strong	0.32	0.32	0.65	0.65
1.00	Strong	0.65	0.65	0.65	0.65
4.00	Strong	1.30	1.30	0.65	0.65

Results regarding the factor loadings. Table 2 provides results of the investigation of how data characteristics influenced the sizes of factor loadings. These investigations were conducted separately for the non-standardized and standardized factor loadings. The first column of Table 2 lists dispersion levels and the second one the source levels. Instead of also listing the different numbers of manifest variables, the results are presented separately for the two levels. Mean non-standardized factor loadings based on 10 and 20 manifest variables are reported in the third and fourth columns, respectively. The fifth and sixth columns include the corresponding standardized factor loadings.

There were three noteworthy observations: (1) the independent variable dispersion showed some degree of covariation with the *non-standardized* factor loadings of 10 and 20 manifest variables but no covariation with standardized factor loadings. (2) The control variable that was influence of source showed covariation with non-standardized and standardized factor loadings. But, it was larger in standardized than in non-standardized ones. (3) The standardized factor loadings virtually corresponded to the expected factor loadings selected for data generation (see method section).

Discussion

Factor loadings and factor variances estimated in conducting CFA and SEM (Brown, 2015; Kline, 2016) provide information on the amount of systematic variation of data captured by the factors of the measurement model. This information can be considered complementary to the information on model fit (DiStefano, 2016; Hu & Bentler, 1999). While model fit informs about the completeness in capturing the systematic variation of data, factor loadings and factor variance focus the amount of captured systematic variation. A particular property of factor loadings and factor variances is that they provide information on parts of a complex model structure. For example, there may be several trait and method factors of a multitrait-multimethod model (Byrne, 2016), an item-position model (Zeller, Reis, & Schweizer, 2017) or a speed-effect model (Ren, Wang, Sun, Deng, & Schweizer, 2017) where information on the contributions of individual factors is of particular interest.

The interdependency of parameters of the covariance model has hampered the consideration of the variance parameter as a source of information. The recent availability of several scaling methods and analyses of their properties (Little et al., 2006; Klopp & Klößner, 2020; Schweizer, 2011; Schweizer et al., 2019) has changed the situation since it has become possible to take control of this dependency. Furthermore, since factor-analytic methods have become available for decomposing systematic variation into parts representing different systematic influences, it is possible and informative to learn about the sizes of the parts captured by factors. For example, it might be interesting to compare the sizes of systematic variation due to traits and method effects (Campbell & Fiske, 1959). Various method effects can potentially impair measurement and need to be controlled in investigations of the validity of data (Kubinger, 2008; Maul, 2013; Schweizer, 2020).

In the simulation study, we investigated the effects of different levels of dispersion that could be interpreted as levels of squeezing or stretching (Vogt & Johnson, 2015) on the factor variance. The estimated variances showed the expected changes based on the analyses of the properties of variances. Although the effect of dispersion was only a simulated effect observed under restricting assumptions, the results made clear that the influence of date dispersion on factor variances is an important issue.

The effect of the number of manifest variables on the factor variance was no surprise, as it was in agreement with the definition of factor variances as sums of squared factor loadings. The increase in factor variance differed from the increase in reliability predicted by classical test theory (Johnson & Morgen, 2016).

A limitation of the reported study is the assumptions that manifest variables do not differ in their dispersion and factor loadings. Although in real data manifest variables are unlikely to display the same degree of variation regarding dispersion and factor loadings, we do not expect that investigations using real data will lead to entirely different results. Another limitation is that effects of non-normality of data in parameter estimation that influence the variance parameter (Schweizer, DiStefano, & French, 2023) are ignored. Furthermore, performance under mis-specifications of models that can provide additional important information (Themessl-Huber, 2014) was not considered. But, it is up to further research to create more flexible predictions of the effects of dispersion and the number of manifest variables.

In brief, factor variances are statistics that provide information on the amount of systematic variation for which factors account. However, factor variances not only reflect systematic variation but also show various dependencies, such as data dispersion, the number of manifest variables, and the underlying structure including the strength of relation between manifest and latent variables. These dependencies must be taken into consideration when interpreting factor variances. The standardization of factor loadings can help controlling for these dependencies.

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